

On the Theory of Elliptically Contoured Distributions

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The theory of elliptically contoured distributions is presented in an unrestricted setting, with no moment restrictions or assumptions of absolute continuity. These distributions are defined parametrically through their characteristic functions and then studied primarily through the use of stochastic representations which naturally follow from the work of Schoenberg [5] on spherically symmetric distributions. It is shown that the conditional distributions of elliptically contoured distributions are elliptically contoured, and the conditional distributions are precisely identified. In addition, a number of the properties of normal distributions (which constitute a type of elliptically contoured distributions) are shown, in fact, to characterize normality.

1. INTRODUCTION

The elliptically contoured distributions on the n -dimensional Euclidean space \mathbb{R}^n are defined as follows. If X is an n -dimensional random (row) vector and, for some $\mu \in \mathbb{R}^n$ and some $n \times n$ nonnegative definite matrix Σ , the characteristic function $\phi_{X-\mu}(t)$ of $X-\mu$ is a function of the quadratic form $t\Sigma t'$, $\phi_{X-\mu}(t) = \phi(t\Sigma t')$, we say that X has an elliptically contoured distribution with parameters μ , Σ and ϕ , and we write $X \sim EC_n(\mu, \Sigma, \phi)$. When $\phi(u) = \exp(-u/2)$, $EC_n(\mu, \Sigma, \phi)$ is the normal distribution $N_n(\mu, \Sigma)$; and when $n = 1$, the class of elliptically contoured distributions coincides with the class of one-dimensional symmetric distributions.

Several properties of elliptically contoured distributions have been

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obtained by Kelker [3] and by Das Gupta *et al.* [1] when Σ is invertible and a density exists. Here, we consider the general case of elliptically contoured distributions, and by making use of a convenient stochastic representation which follows from the work of Schoenberg [5], we show that their conditional distributions are also elliptically contoured and permit an intuitively appealing stochastic representation, and we derive several new characterizations of the normal distributions. Section 2 develops the basic properties of the stochastic and parametric representations; Section 3 evaluates the conditional distributions; Section 4 discusses densities; and Section 5 is devoted to characterizations of normality.

Throughout, $X \stackrel{d}{=} Y$ denotes that the random vectors X and Y are identically distributed, I_k denotes the $k \times k$ identity matrix, $r(A)$ denotes the rank of the matrix A , and A^- denotes a generalized inverse.

2. STOCHASTIC AND PARAMETRIC REPRESENTATIONS

In this section we develop the basic properties of the stochastic and parametric representations of elliptically contoured distributions.

The stochastic representations are based on the following results of Schoenberg [5]. If Φ_k , $k \geq 1$, is the class of all functions $\phi: [0, \infty) \rightarrow \mathbb{R}$ such that $\phi(\|t\|^2)$, $t \in \mathbb{R}^k$, is a characteristic function, then $\phi \in \Phi_k$ if and only if

$$\phi(u) = \int_{[0, \infty)} \Omega_k(r^2 u) dF(r), \quad u \geq 0, \quad (1)$$

for some distribution function (d.f.) F on $[0, \infty)$, where $\Omega_k(\|t\|^2)$, $t \in \mathbb{R}^k$, is the characteristic function (ch.f.) of a k -dimensional random vector $U^{(k)}$ which is uniformly distributed on the unit sphere in \mathbb{R}^k . Also $\Phi_k \downarrow \Phi_\infty$, and $\phi \in \Phi_\infty$ if and only if ϕ is given as in (1) with $\Omega_k(r^2 u)$ replaced by $\exp(-r^2 u/2)$. The notation Φ_k , $U^{(k)}$, Ω_k will be used throughout.

THEOREM 1. $X \sim EC_n(\mu, \Sigma, \phi)$ with $r(\Sigma) = k$ if and only if

$$X \stackrel{d}{=} \mu + RU^{(k)}A, \quad (2)$$

where $R \geq 0$ is independent of $U^{(k)}$, $\Sigma = A'A$ is a rank factorization of Σ ($A: k \times n$, $r(A) = k$), and the distribution function F of R is related to ϕ as in (1).

Proof. The "if" part follows immediately from (1). For the "only if" part note that $Y = (X - \mu)A^-$ has characteristic function $\phi(\|t\|^2)$, $t \in \mathbb{R}^k$; hence $\phi \in \Phi_k$, and by (1), $Y \stackrel{d}{=} RU^{(k)}$. ■

It is clear from Theorem 1 that the class of admissible functions ϕ in the parametric representation $EC_n(\mu, \Sigma, \phi)$ with $r(\Sigma) = k$ is Φ_k . We shall refer to the stochastic representation given in (2) as a *full rank representation* of X . It is to be distinguished from a similar representation of $X \sim EC_n(\mu, \Sigma, \phi)$ with $r(\Sigma) = k$ of the form

$$X \stackrel{d}{=} \mu + RU^{(l)}A, \quad (3)$$

where $R \geq 0$ and $U^{(l)}$ are independent, $\Sigma = A'A$ is not necessarily a rank factorization of Σ , and the distribution function F of R is related to ϕ as in (1) with k replaced by l ; it is easy to show that X is representable as in (3) if and only if $l \geq k$ and $\phi \in \Phi_l (\subset \Phi_k)$. For instance, if $X \sim EC_n(\mu, \Sigma, \phi)$ with full rank representation $X \stackrel{d}{=} \mu + RU^{(k)}A$, and if $c \in \mathbb{R}^m$ and $B: n \times m$, then $XB + c \stackrel{d}{=} (\mu B + c) + RU^{(k)}(AB)$, which is a full rank representation if $r(AB) = k$ and is non-full-rank if $r(AB) < k$.

When Σ is of full rank n , then $X \sim EC_n(\mu, \Sigma, \phi)$ has a full rank representation taking the form

$$X \stackrel{d}{=} \mu + RU^{(n)}\Sigma^{1/2}. \quad (4)$$

The special significance of the distribution function F appearing in (1) with $k = r(\Sigma)$ is made clear in the following corollary:

COROLLARY 1. *With the notation of Theorem 1, the quadratic form*

$$Q(X) = (X - \mu)\Sigma^{-1}(X - \mu)' \quad (5)$$

has the distribution function $F(\sqrt{\cdot})$.

Proof. From the full rank representation (2) and the properties of generalized inverses one obtains $Q(X) \stackrel{d}{=} R^2 U^{(k)}A(A'A)^{-1}A'U^{(k)} = R^2$. ■

We now show that the relationship (1) between $\phi \in \Phi_k$ and F is one-to-one, and, in fact, we provide some kind of an inverse.

THEOREM 2. $\phi \in \Phi_k$ if and only if ϕ is continuous and

$$\int_0^\infty \phi(2sv) g_k(v) dv, \quad s \geq 0, \quad (6)$$

is the Laplace transform of a nonnegative random variable (namely, $Q(X)$ given by (5)) whose distribution function is $F(\sqrt{\cdot})$ when ϕ is given by (1) where g_k denotes the chi-squared density with k degrees of freedom ($1 \leq k < \infty$).

Proof. Since $N_k(0, I_k)$ is $EC_k(0, I_k, \phi)$ with $\phi(u) = \exp(-u/2)$, (1) yields the identity

$$\exp(-u/2) = \int_0^\infty \Omega_k(uv) g_k(v) dv, \quad u \geq 0, \quad (7)$$

where g_k is the density of the quadratic form $Q(X) = XX'$ (see Corollary 1). When $\phi \in \Phi_k$, it follows immediately from (1) and (7) that (6) is the Laplace transform of R^2 where R has the d.f. F . Conversely, suppose (6) is the Laplace transform of R^2 and ϕ is continuous. Let F be the d.f. of R , and let $\phi_0 \in \Phi_k$ be the function defined as in (1). From the first part of the proof we have

$$\int_0^\infty \phi_0(2sv) g_k(v) dv = \int_0^\infty \phi(2sv) g_k(v) dv, \quad s \geq 0,$$

and since $g_k(v) = (\text{const}) v^{k/2-1} \exp(-v/2)$, the uniqueness of Laplace transforms implies that $\phi_0 = \phi$. ■

Theorem 2, together with Bernstein's theorem (cf. Feller [2, p. 439]), can be used effectively to check whether a function ϕ belongs to the class Φ_k . For instance, it is easy to show that $\phi(\cdot) = (1 - \alpha \cdot) \exp(-\cdot/2) \in \Phi_k$ if and only if $0 \leq \alpha k \leq 1$ ($1 \leq k < \infty$).

The following lemmas will be needed in the sequel.

LEMMA 1. Suppose $X = {}^d R U^{(n)} \sim EC_n(0, I_n, \phi)$ and $P(X=0) = 0$. Then

$$\|X\| \stackrel{d}{=} R, \quad X/\|X\| \stackrel{d}{=} U^{(n)},$$

and they are independent.

Proof. Since the mapping $x \rightarrow (\|x\|, x/\|x\|)$ is Borel measurable on $\mathbb{R}^n - \{0\}$, it follows from the representation $X = {}^d R U^{(n)}$ that $(\|X\|, X/\|X\|) = {}^d (R, U^{(n)})$, which proves the lemma. ■

Write $U^{(n)} = (U_1^{(n)}, U_2^{(n)})$, where $U_1^{(n)}$ is m -dimensional ($1 \leq m < n$).

LEMMA 2. $(U_1^{(n)}, U_2^{(n)}) = {}^d (R_{mn} U^{(m)}, (1 - R_{mn}^2)^{1/2} U^{(n-m)})$, where $R_{mn} (\geq 0)$, $U^{(m)}$ and $U^{(n-m)}$ are independent, and $R_{mn}^2 \sim \text{Beta}(m/2, (n-m)/2)$.

Proof. Let $X = (X_1, X_2) \sim N_n(0, I_n)$, where the dimension of X_1 is m . Since X_1 and X_2 are independent, it follows from Lemma 1 that

$$X_1/\|X_1\| = U^{(m)}, \quad X_2/\|X_2\| = U^{(n-m)}, \quad \|X_1\|, \quad \|X_2\|$$

are jointly independent. If we define R_{mn} as $\|X_1\|/\|X\|$, then R_{mn} , $U^{(m)}$ and

$U^{(n-m)}$ are independent, and R_{mn} has the required distribution since $\|X_1\|^2$ and $\|X_2\|^2$ are independent and chi-squared with degrees of freedom m and $n-m$, respectively. Finally, applying Lemma 1 again, we obtain

$$\begin{aligned}(U_1^{(n)}, U_2^{(n)}) &= U^{(n)} \stackrel{d}{=} \frac{X}{\|X\|} = \left(\frac{X_1}{\|X\|}, \frac{X_2}{\|X\|} \right) \\ &= (R_{mn} U^{(m)}, (1 - R_{mn}^2)^{1/2} U^{(n-m)}). \quad \blacksquare\end{aligned}$$

By Theorem 2, the relationship (1) between $\phi \in \Phi_k$ and F is one-to-one. However, if $\phi \in \Phi_k$, then for each $1 \leq n \leq k$, $\phi \in \Phi_n$, and according to (1) and Theorem 2,

$$\phi(u) = \int_{[0, \infty)} \Omega_n(r^2 u) dF_n(r), \quad u \geq 0,$$

for some unique distribution function F_n on $[0, \infty)$. The precise relationship between the various F_n 's, $1 \leq n \leq k$, is specified in the following:

COROLLARY 2. *If $1 \leq m < n \leq k$ and R_m, R_n have distribution functions F_m, F_n , then $R_m \stackrel{d}{=} R_{mn} R_n$, where $R_{mn}^2 \sim \text{Beta}(m/2, (n-m)/2)$ is independent of R_n ; i.e., F_m has an atom of size $F_n(0)$ at zero, and it is absolutely continuous on $(0, \infty)$ with the density*

$$f_m(s) = \frac{2s^{m-1} \Gamma(n/2)}{\Gamma(m/2) \Gamma((n-m)/2)} \int_s^\infty r^{-(n-2)} (r^2 - s^2)^{(n-m)/2-1} dF_n(r),$$

$$0 < s < \infty. \quad (8)$$

Proof. Let $X = (X_1, X_2) \stackrel{d}{=} R_n U^{(n)} \sim EC_n(0, I_n, \phi)$, where X_1 is of dimension m . Then $X_1 \stackrel{d}{=} R_m U^{(m)} \sim EC_m(0, I_m, \phi)$. But, on the other hand, we have from Lemma 2, $X_1 \stackrel{d}{=} R_n U_1^{(n)} \stackrel{d}{=} R_{mn} R_n U^{(m)}$, where R_n, R_{mn} and $U^{(m)}$ are independent. Thus, X_1 has two full rank representations, $R_m U^{(m)}$ and $R_n R_{mn} U^{(m)}$, and the one-to-one correspondence between ϕ and F in (1), established in Theorem 2, implies $R_m \stackrel{d}{=} R_{mn} R_n$. \blacksquare

We now show that the stochastic, as well as the parametric, representations of nondegenerate elliptically contoured distributions are essentially unique, i.e., up to only the obvious redundancies.

THEOREM 3. *Let X be nondegenerate.*

(i) *If $X \sim EC_n(\mu, \Sigma, \phi)$ and $X \sim EC_n(\mu^*, \Sigma^*, \phi^*)$, then there is a $c > 0$ such that*

$$\mu^* = \mu, \quad \Sigma^* = c\Sigma, \quad \phi^*(\cdot) = \phi(c^{-1}\cdot). \quad (9)$$

(ii) If $X = {}^d\mu + RU^{(l)}A$ and $X = {}^d\mu^* + R^*U^{(l^*)}A^*$, where $l \geq l^*$, then there is a $c > 0$ such that

$$\mu^* = \mu, \quad A^{*'}A^* = cA'A, \quad R^* \stackrel{d}{=} c^{-1/2}R_{l^*,l}R$$

where $R_{l^*,l}$ is independent of R and $R_{l^*,l} \sim \text{Beta}(l^*/2, (l - l^*)/2)$ if $l > l^*$, and $R_{l^*,l} = 1$ if $l = l^*$.

Proof. (i) Since $X - \mu$ and $X - \mu^*$ are symmetric about 0, we have $X - \mu \stackrel{d}{=} -(X - \mu) = \mu - \mu^* - (X - \mu^*) \stackrel{d}{=} \mu - \mu^* + (X - \mu^*) = X - (2\mu^* - \mu)$, which implies $\mu^* = \mu$. Write $\Sigma = (\sigma_{ij})$, $\Sigma^* = (\sigma_{ij}^*)$, $\mu = (\mu_i)$. Since $X = (X_i)$ is not degenerate, then one of its components X_j is not degenerate, and the ch.f. of $X_j - \mu_j$ is given by $\phi(\sigma_{jj}u^2) = \phi^*(\sigma_{jj}^*u^2)$, $u \in \mathbb{R}$, with $\sigma_{jj}, \sigma_{jj}^* > 0$, which establishes $\phi^*(\cdot) = \phi(c^{-1}\cdot)$ with $c = \sigma_{jj}^*/\sigma_{jj}$. Then the ch.f. of $X - \mu$ is given by $\phi(t\Sigma t') = \phi^*(t\Sigma^*t') = \phi(c^{-1}t\Sigma^*t)$, $t \in \mathbb{R}^n$. Now if $\Sigma^* \neq c\Sigma$, then for some $t_0 \in \mathbb{R}^n$, $a^2 = t_0\Sigma t_0' \neq c^{-1}t_0\Sigma^*t_0' = b^2$, and putting $t = ut_0$, we obtain $\phi(a^2u^2) = \phi(b^2u^2)$, $u \in \mathbb{R}$. But this implies through recursion $\phi(u^2) = \phi(0) = 1$, $u \in \mathbb{R}$, which contradicts the nondegeneracy of X ; and thus $\Sigma^* = c\Sigma$ follows.

(ii) The assumptions imply that $X \sim EC_n(\mu, A'A, \phi)$ and $X \sim EC_n(\mu^*, A^{*'}A^*, \phi^*)$; so by (i), $\mu^* = \mu$, $A^{*'}A^* = cA'A$, $\phi^*(\cdot) = \phi(c^{-1}\cdot)$. Thus, putting $R_l = R$, $R_{l^*} = c^{1/2}R^*$, $\Sigma = A'A$, we have $X - \mu \stackrel{d}{=} R_l U^{(l)}A \sim EC_n(0, \Sigma, \phi)$, $\phi \in \Phi_l$, and also $X - \mu \stackrel{d}{=} R_{l^*} U^{(l^*)}(c^{-1/2}A^*) \sim EC_n(0, \Sigma, \phi)$, $\phi \in \Phi_{l^*}$. When $l^* < l$, it follows from Corollary 2 with $m = l^*$ and $n = l$ that $R_{l^*} \stackrel{d}{=} R_{l^*,l}R_l$ and thus $R^* \stackrel{d}{=} c^{-1/2}R_{l^*,l}R$. When $l^* = l$, the one-to-one correspondence between ϕ and F in (1) (cf. Theorem 2) implies $R_{l^*} \stackrel{d}{=} R_l$ and, thus, $R^* \stackrel{d}{=} c^{-1/2}R$. ■

It follows from Theorem 3 that if $X \sim EC_n(\mu, \Sigma, \phi)$ has the representation (3) and is nondegenerate, then X has a full rank representation $X \stackrel{d}{=} \mu + RR_{kl}U^{(k)}B$ and the quadratic form defined by (5) has representation $Q(X) \stackrel{d}{=} R^2R_{kl}^2$, where $k = r(\Sigma)$, $B'B$ is a rank factorization of $\Sigma = A'A$, and $R, R_{kl}, U^{(k)}$ are independent with R_{kl} distributed as in (ii).

If $X \sim EC_n(\mu, \Sigma, \phi)$, it is clear from (2) that all components of X have finite first moment if and only if $\mathcal{E}R < \infty$, and that in this case the location parameter μ is the mean of X . Also (excluding the degenerate case $\Sigma = 0$) all components of X have finite second moment if and only if $\mathcal{E}R^2 < \infty$, and in this case the scale parameter Σ is proportional to the covariance matrix of X , $\mathcal{E}(X - \mu)'(X - \mu) = \mathcal{E}R^2 \cdot A'U^{(k)'}U^{(k)}A = k^{-1}\mathcal{E}R^2 \cdot \Sigma$. An alternative characterization and expression of the covariance matrix, in terms of ϕ , is given in the following:

THEOREM 4. Let $X \sim EC_n(\mu, \Sigma, \phi)$ be nondegenerate. The covariance

matrix Σ_0 of X exists if and only if the right-hand derivative of $\phi(u)$ at $u = 0$, denoted by $\phi'(0)$, exists and is finite, in which case $\Sigma_0 = -2\phi'(0)\Sigma$.

Proof. Σ_0 exists iff $\mathcal{E}R^2 < \infty$ or iff $\mathcal{E}X_1^2 < \infty$. Since the ch.f. of X_1 is $\phi_{X_1}(t) = \phi(t^2)$, $t \in \mathbb{R}$, the result follows from the properties of ch.f.'s of symmetric r.v.'s (cf. Lukacs [4]). ■

When the covariance matrix Σ_0 of $X \sim EC_n(\mu, \Sigma, \phi)$ exists, choosing Σ to be Σ_0 has special appeal. This occurs if and only if ϕ is chosen so as to make $-2\phi'(0) = 1$; e.g., in the case of normality, among the possibilities $\{\exp(-cu), c > 0\}$, one chooses $\phi(u) = \exp(-u/2)$.

3. CONDITIONAL DISTRIBUTIONS

In this section, it is shown that if two random vectors have a joint elliptically contoured distribution, then the conditional distribution of one given the other is also elliptically contoured. The location and scale parameters of the conditional distribution do not depend upon the third parameter ϕ of the joint distribution, and consequently, the formulas which apply in the normal case apply in this more general setting as well. The case in which the scale parameter Σ of the joint distribution is an identity matrix is considered first in Theorem 5, and the general case is handled in Corollary 5. The following lemma, whose proof is straightforward and therefore omitted, is needed in the proof of Theorem 5.

LEMMA 3. *Suppose the nonnegative random variables R and S are independent, R has distribution function F , and S is absolutely continuous with density g . Then $T = RS$ has an atom of size $F(0)$ at zero if $F(0) > 0$, it is absolutely continuous on $(0, \infty)$ with density h given by*

$$h(t) = \int_{(0, \infty)} r^{-1} g(t/r) dF(r),$$

and a regular version of the conditional distribution of R given $T = t$ is expressible as

$$\begin{aligned} P(R \leq \rho \mid T = t) &= 0 && \text{for } \rho < 0; \\ &= 1 && \text{for } t = 0 \text{ or } t > 0 \text{ with } h(t) = 0, \rho \geq 0; \\ &= \frac{1}{h(t)} \int_{(0, \rho]} r^{-1} g(t/r) dF(r) && (10) \\ &&& \text{for } t > 0 \text{ with } h(t) \neq 0, \rho \geq 0. \end{aligned}$$

THEOREM 5. Let $X = RU^{(n)} \sim EC_n(0, I_n, \phi)$ and $X = (X_1, X_2)$, where X_1 is m -dimensional, $1 \leq m < n$. Then a regular conditional distribution of X_1 given $X_2 = x_2$ is given by

$$(X_1 | X_2 = x_2) \sim EC_m(0, I_m, \phi_{\|x_2\|^2}) \quad (11)$$

with full rank representation

$$(X_1 | X_2 = x_2) \stackrel{d}{=} R_{\|x_2\|^2} U^{(m)}, \quad (12)$$

where for each $a \geq 0$, R_{a^2} and $U^{(m)}$ are independent, the distribution of R_{a^2} is given by (15) (below),

$$R_{\|x_2\|^2} \stackrel{d}{=} ((R^2 - \|x_2\|^2)^{1/2} | X_2 = x_2), \quad (13)$$

and the function ϕ_{a^2} is given by (1) with $k = m$ and F replaced by the distribution function of R_{a^2} .

Proof. From Lemma 2 we have

$$(X_1, X_2) = R(U_1^{(n)}, U_2^{(n)}) \stackrel{d}{=} R(R_{mn} U^{(m)}, (1 - R_{mn}^2)^{1/2} U^{(n-m)}),$$

where R , R_{mn} , $U^{(m)}$ and $U^{(n-m)}$ are independent. Since $RR_{mn} = (R^2 - \|x_2\|^2)^{1/2}$ when $R(1 - R_{mn}^2)^{1/2} U^{(n-m)} = x_2$, (12) and (13) follow. That $R_{\|x_2\|^2}$, as defined in (13), depends upon x_2 only through $\|x_2\|^2$, as the notation indicates, is obvious when $x_2 = 0$; and when $x_2 \neq 0$, it follows from

$$\begin{aligned} ((R^2 - \|x_2\|)^{1/2} | X_2 = x_2) &\stackrel{d}{=} ((R^2 - \|x_2\|^2)^{1/2} | R(1 - R_{mn}^2)^{1/2} U^{(n-m)} = x_2) \\ &\stackrel{d}{=} ((R^2 - \|x_2\|^2)^{1/2} | R^2(1 - R_{mn}^2) = \|x_2\|^2) \end{aligned} \quad (14)$$

since $U^{(n-m)}$ is independent of R and R_{mn} . Clearly (11) follows from (2) with $\phi_{\|x_2\|^2}$ defined in the manner described. Finally from (13), (14), and Lemma 3, with the density of $(1 - R_{mn}^2)^{1/2}$ given by (cf. Lemma 2) $g(t) = \Gamma(m/2) \Gamma((n-m)/2) \Gamma^{-1}(n/2) t^{n-m-1} (1-t^2)^{(m/2)-1}$, $0 < t < 1$, and the d.f. of R given by F , one obtains

$$\begin{aligned} R_{a^2} &= 0 \text{ a.s. when } a = 0 \text{ or } F(a) = 1, \\ P(R_{a^2} \leq \rho) &= \frac{\int_{(a, \sqrt{\rho^2 + a^2})} (r^2 - a^2)^{m/2-1} r^{-(n-2)} dF(r)}{\int_{(a, \infty)} (r^2 - a^2)^{m/2-1} r^{-(n-2)} dF(r)}, \end{aligned} \quad (15)$$

$\rho \geq 0$, $a > 0$ and $F(a) < 1$. ■

While (15) shows how the distribution function F_{a^2} of R_{a^2} is determined by F (together with m , n and a), in the converse direction we have the following which is germane to Theorem 7 below: Denoting the denominator in (15) by C_{a^2} , we obtain from (14)

$$1 - F(r) = C_{a^2} \int_{(\sqrt{r^2 - a^2}, \infty)} (\rho^2 + a^2)^{n/2 - 1} \rho^{-(m-2)} dF_{a^2}(\rho), \quad r \geq a, \quad (16)$$

which is valid for all $a > 0$. Thus, when $a > 0$, F_{a^2} determines F on the interval $[a, \infty)$ up to the unknown multiplicative factor $C_{a^2} \geq 0$, and, of course, it contains no information about the values of F on $[0, a)$. If $F(r)$ is known for some $r \geq a$ and $F(r) < 1$, then F_{a^2} determines F uniquely on $[a, \infty)$ by means of (16).

COROLLARY 5. *Let $X = \mu + RU^{(k)}A \sim EC_n(\mu, \Sigma, \phi)$ with $A'A = \Sigma$ and $r(A) = r(\Sigma) = k \geq 1$. Further, let*

$$X = (X_1, X_2), \quad \mu = (\mu_1, \mu_2), \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

where the dimensions of X_1 , μ_1 and Σ_{11} are, respectively, m , m and $m \times m$, and assume $k_2 = r(\Sigma_{22}) \geq 1$ and $k_1 = k - k_2 \geq 1$. Finally let S denote the row space of Σ_{22} . Then a regular conditional distribution of X , given $X_2 = x_2$, is given by

$$(X_1 | X_2 = x_2) \sim EC_m(\mu_{x_2}, \Sigma^*, \phi_{q(x_2)}) \quad \text{for } x_2 \in \mu_2 + S, \quad (17a)$$

$$(X_1 | X_2 = x_2) \stackrel{d}{=} \mu_1 \quad \text{for } x_2 \notin \mu_2 + S, \quad (17b)$$

with a full rank representation

$$(X_1 | X_2 = x_2) \stackrel{d}{=} \mu_{x_2} + R_{q(x_2)} U^{(k_1)} A^* \quad \text{for } x_2 \in \mu_2 + S, \quad (18)$$

where

$$\mu_{x_2} = \mu_1 + (x_2 - \mu_2) \Sigma_{22}^- \Sigma_{21}, \quad \Sigma^* = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^- \Sigma_{21}, \\ q(x_2) = (x_2 - \mu_2) \Sigma_{22}^- (x_2 - \mu_2)',$$

and $\Sigma^* = A^* A^*$ is a full rank (k_1) factorization. Moreover, for each $a \geq 0$, R_{a^2} is independent of $U^{(k_1)}$ and its distribution is given by (15) with $n = k$ and $m = k_1$;

$$R_{q(x_2)} \stackrel{d}{=} ((R^2 - q(x_2))^{1/2} | X_2 = x_2) \quad \text{for } x_2 \in \mu_2 + S,$$

and the function ϕ_{a^2} is given by (1) with $k=k_1$ and F replaced by the distribution function of R_{a^2} .

Proof. Write $RU^{(k)} = (Z_1, Z_2)$, where Z_1 is k_1 -dimensional, and let $\Sigma_{22} = A'_2 A_2$ be a rank factorization of Σ_{22} so that A_2 is $k_2 \times (n-m)$ and $r(A_2) = k_2$. It is easily checked through their characteristic functions that

$$(X_1, X_2) \stackrel{d}{=} (\mu_1 + Z_1 A^* + Z_2 A_2 \Sigma_{22}^- \Sigma_{21}, \mu_2 + Z_2 A_2).$$

Thus, $(X_1 | X_2 = x_2) \stackrel{d}{=} \mu_{x_2} + (Z_1 | Z_2 A_2 = x_2 - \mu_2) A^*$, and the proof is completed in a straightforward way using $(Z_1 | Z_2 = z_2)$ from Theorem 5. ■

The description of $(X_1 | X_2 = x_2)$ given in (17a) is of primary interest since $X_2 \in \mu_2 + S$ a.s. Also the excluded cases $k_2 = 0$ and $k_1 = 0$ are trivial since if $k_2 = 0$, then $X_2 = \mu_2$ a.s., and if $k_1 = 0$, then $X_1 = \mu_{x_2} = \mu_1 + (X_2 - \mu_2) \Sigma_{22}^- \Sigma_{21}$ a.s.

It is not possible to express ϕ_{a^2} directly in terms of ϕ , but $\phi_{a^2}(u)$ is expressed in terms of a , u , and F by (1), with $k=k_1$ and F replaced by the distribution function of R_{a^2} given in (15). For simpler expressions in special cases see Theorems 6 and 9.

Also, unlike the normal case (with $\phi(u) = \exp(-u/2)$), if Σ is the covariance matrix of X , Σ^* need not be the conditional covariance matrix of X_1 given X_2 , which, whenever it exists, equals $k_1^{-1} \mathcal{E} R_{q(x_2)}^2 \Sigma^*$. The conditional covariance matrix of X_1 given X_2 exists if and only if $\mathcal{E} R_a^2 < \infty$, and it is easily seen by (15) that this is satisfied if $k_1 \leq k-2$, or if $\mathcal{E} R < \infty$ and $k_1 \leq k-1$, or, of course, if $\mathcal{E} R^2 < \infty$.

4. DENSITIES

Here we briefly discuss the densities of elliptically contoured distributions, and we introduce notation which will be needed in Section 5. Much of this material appears elsewhere, such as in Kelker [3], with a somewhat different emphasis.

If $X \sim EC_n(0, I_n, \phi)$ is absolutely continuous, its density is invariant under all orthogonal transformations and, thus, is expressible in terms of a function $g_n: [0, \infty) \rightarrow [0, \infty)$ as

$$g_n(x_1^2 + \cdots + x_n^2), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n. \quad (19)$$

Moreover, X has a full rank representation of the form $R_n U^{(n)}$, where R_n is also absolutely continuous, and its density f_n is expressible as

$$f_n(r) = s_n r^{n-1} g_n(r^2), \quad r \geq 0, \quad (20)$$

where s_n denotes the surface area of the unit ball in \mathbb{R}^n ($s_1 = 1$). Thus, g_n must satisfy the integrability condition $\int_0^\infty r^{n-1} g_n(r^2) dr < \infty$. Conversely, if R is absolutely continuous, then so is X , and their densities are related through (20).

If an absolutely continuous random vector $X = (X_1, \dots, X_n)$ has a density of the form shown in (19), then f_n , defined by (20), is a density of a nonnegative random variable R , and $X \sim EC_n(0, I_n, \phi)$, where ϕ is defined by (1) with $k = n$ and $dF(x) = f_n(x) dx$.

Let $Y \sim EC_m(0, I_m, \phi)$ denote an arbitrary marginal of X of dimension m ($1 \leq m < n$). Y has the canonical representation $R_m R_n U^{(m)}$ (cf. Lemma 2) and is absolutely continuous if and only if $P(R_n = 0) = 0$. When this occurs, the density f_m of R_m is given in (8), where F_n denotes the distribution function of R_n , and, thus, the density of Y takes the form $g_m(\|y\|^2)$, $y \in \mathbb{R}^m$, where

$$g_m(u) = s_n^{-1} s_{(n-m)} \int_u^\infty r^{-(n-2)} (r^2 - u)^{(n-m)/2-1} dF_n(r). \quad (21)$$

When, moreover, R_n is absolutely continuous, this simplifies to

$$g_m(u) = s_{(n-m)} \int_0^\infty g_n(u + r^2) r^{n-m-1} dr. \quad (22)$$

More generally, if $X \sim EC_n(\mu, \Sigma, \phi)$ with Σ of full rank n , then $(X - \mu) \Sigma^{-1/2} \sim EC_n(0, I_n, \phi)$. Thus, the density of X exists and assumes the form

$$|\Sigma|^{-1/2} g_n((x - \mu) \Sigma^{-1} (x - \mu)) \quad (23)$$

if and only if R , appearing in the full rank representation $\mu + R U^{(n)} \Sigma^{1/2}$, is absolutely continuous with the density f_n defined in (20). When Σ is of rank n and X is absolutely continuous, in terms of the notation of Corollary 5, the conditional density of X_1 given $X_2 = x_2$ is given by

$$\frac{\left(g_n([x_1 - \mu_1 - (x_2 - \mu_2) \Sigma_{22}^{-1} \Sigma_{21}] \Sigma^{*-1} [x_1 - \mu_1 - (x_2 - \mu_2) \Sigma_{22}^{-1} \Sigma_{21}]' + (x_2 - \mu_2) \Sigma_{22}^{-1} (x_2 - \mu_2)') \right)}{s_m |\Sigma^*|^{1/2} \int_0^\infty g_n(r^2 + (x_2 - \mu_2) \Sigma_{22}^{-1} (x_2 - \mu_2)') r^{m-1} dr}.$$

Finally, suppose $X \sim EC_n(\mu, \Sigma, \phi)$ with $r(\Sigma) = n$ and $\phi \in \Phi_\infty$. By the characterization of Φ_∞ (preceding Theorem 1), there exists a distribution function F_∞ such that

$$\phi(u) = \int_{[0, \infty)} \exp(-ur^2/2) dF_\infty(r), \quad u \geq 0.$$

Since $\phi(t\Sigma t')$, $t \in \mathbb{R}^n$, is the characteristic function of $X - \mu$, X is absolutely continuous and has the density (23) with

$$g_n(u) = \int_{(0, \infty)} (2\pi r^2)^{-n/2} \exp(-u/2r^2) dF_\infty(r) \quad (24)$$

if and only if $F_\infty(0) = 0$. Otherwise, X is absolutely continuous away from μ with density given in (23) and (24) and has an atom of size $F_\infty(0)$ at μ .

5. CHARACTERIZATIONS OF NORMALITY

In this section we focus attention on several properties of the normal distributions which do not extend to other elliptically contoured distributions. One such characterization of normality is an immediate consequence of Corollary 1 and Theorem 2: $X \sim EC_n(\mu, \Sigma, \phi)$ is *nondegenerate normal* if and only if $Q(X)$ (defined in (5)) is a *positive multiple of a chi-squared random variable with $k = r(\Sigma)$ degrees of freedom*.

(a) When $X = (X_1, X_2)$ is normally distributed, then, of course, the conditional distribution of X_1 , given X_2 , is normal and the function $\phi_{q(x_2)}$ in Corollary 5 assumes the form $\phi_{q(x_2)} = \exp(-cu/2)$, where $c \geq 0$ is independent of x_2 . The failure of $\phi_{q(x_2)}$ to depend upon the value of $q(x_2)$ characterizes normality:

THEOREM 6. *Assume the rank values k_1 and k_2 appearing in Corollary 5 are strictly positive. Then $\phi_{q(x_2)}(u)$ is degenerate for each $u \geq 0$ if and only if X is normally distributed.*

Proof. It suffices to show the "only if" part. By Corollary 5, for all $t = (t_1, t_2) \in \mathbb{R}^n$,

$$\begin{aligned} \phi(t\Sigma t') &= \mathcal{E} \exp[it(X - \mu)'] \\ &= \mathcal{E} \{ \exp[it_2(X_2 - \mu_2)'] \mid \mathcal{E}(\exp[it_1(X_1 - \mu_1)'] \mid X_2) \} \\ &= \mathcal{E} \{ \exp[it_2(X_2 - \mu_2)' + it_1(\mu_{X_2} - \mu_1)'] \phi_{q(X_2)}(t_1 \Sigma^* t_1) \}. \end{aligned}$$

Hence, putting, for each $u \geq 0$, $\phi_{q(X_2)}(u) = \psi(u)$ a.s., it follows that

$$\begin{aligned} \phi(t\Sigma t') &= \psi(t_1 \Sigma^* t_1') \mathcal{E} \exp[i(t_2 + t_1 \Sigma_{12} \Sigma_{22}^{-1})(X_2 - \mu_2)'] \\ &= \psi(t_1 \Sigma^* t_1) \phi\{(t_2 + t_1 \Sigma_{12} \Sigma_{22}^{-1}) \Sigma_{22}(t_2 + t_1 \Sigma_{12} \Sigma_{22}^{-1})'\}. \end{aligned}$$

And since $t\Sigma t' = t_1 \Sigma^* t_1' + (t_2 + t_1 \Sigma_{12} \Sigma_{22}^{-1}) \Sigma_{22}(t_2 + t_1 \Sigma_{12} \Sigma_{22}^{-1})'$, it follows that $\phi(u + v) = \psi(u) \phi(v)$, $u, v \geq 0$. (Here, one makes use of the assumption k_1 ,

$k_2 \geq 1$.) Setting $v = 0$ yields $\phi = \psi$, and, thus, $\phi(u + v) = \phi(u)\phi(v)$, $u, v \geq 0$. Since ϕ is continuous with $\phi(0) = 1$ and $|\phi(u)| \leq 1$, it follows that $\phi(u) = \exp(-cu/2)$ for some $c \geq 0$, and, thus, $X \sim N_n(\mu, c\Sigma)$. ■

(b) It is apparent from (16) that the normality of $(X_1 | X_2 = x_2)$ for (some) fixed x_2 does not entail the normality of X , for the distribution function of $R_{\|x\|^2}$ can be that of a chi-variable without F being that of a multiple of a chi-variable. Nevertheless, the following is true:

THEOREM 7. *Assume $X = (X_1, X_2)$ has a nondegenerate elliptically contoured distribution and the rank values k_1 and k_2 , appearing in Corollary 5, are both positive. Then X is normally distributed if and only if, with probability 1, the conditional distribution of X_1 given X_2 is nondegenerate normal.*

Proof. It suffices to show the "if" part. Using the notation in the proof of Corollary 5, if $z_2 A_2 = x_2 - \mu_2$ and $x_2 \in \mu_2 + S$, then $q(x_2) = \|(x_2 - \mu_2) A_2^{-1}\|^2 = \|z_2\|^2$, and, consequently,

$$q(X_2) \stackrel{d}{=} \|Z_2\|^2, \quad (25)$$

where $(Z_1, Z_2) = RU^{(k)}$. Thus, $P(q(X_2) = 0) = P(R = 0) \leq P((X_1 | X_2) \text{ is degenerate}) = 0$, and hence $q(X_2) > 0$ a.s. Also, since $(X_1 | X_2)$ is nondegenerate normal a.s., we have $\phi_{q(X_2)}(u) = \exp(-c(q(X_2))u/2)$, $u \geq 0$, a.s. for some function $c: (0, \infty) \rightarrow (0, \infty)$. If it is shown that $c(q(X_2))$ is a degenerate random variable, then the normality of X will follow from Theorem 6. Let A be the set of all $a > 0$ such that

$$\phi_{a^2}(u) = \exp(-c(a^2)u/2), \quad u \geq 0. \quad (26)$$

Note $P(q(X_2)^{1/2} \in A) = 1$. Now (26) implies that the d.f. F_a of R_{a^2} , $a \in A$, is that of a chi-variable with k_1 degrees of freedom times $c(a^2)^{1/2}$. Combining this with (16), when $n = k$ and $m = k_1$, we have for $a \in A$, $1 - F(r) = (\text{const}) \int_r^\infty s^{k-1} \exp(-s^2/2c(a^2)) ds$, $r \geq a$, i.e.,

$$F'(r) = (\text{const}) r^{k-1} e^{-r^2/2c(a^2)}, \quad r \geq a. \quad (27)$$

It is obvious from (27) that $c(a^2)$ is a constant for all $a \in A$, and the degeneracy of $c(q(X_2))$ follows. ■

An alternative proof, without using Theorem 6, can be given as follows. Equation (26) implies

$$F'(r) = (\text{const}) r^{k-1} e^{-r^2/2c}, \quad r > a_0, \quad (28)$$

where c is some positive constant and a_0 is the infimum of A . Since, by (25),

$P(0 < q(X_2)^{1/2} < \varepsilon) > 0$ for every $\varepsilon > 0$, we have $A \cap (0, \varepsilon) \neq \emptyset$ for every $\varepsilon > 0$, and therefore, $a_0 = 0$. Now it follows from (28) and the fact $F(0) = P(R = 0) = 0$ that F is the distribution of a chi-variable with k degrees of freedom times $c^{1/2}$, and, thus, X is normal. This argument has the advantage that it can be adopted to weaker assumptions: (under the setting of Theorem 7) X is normal if X_1 given $X_2 = x_2$ is nondegenerate normal for all $x_2 \in \mu_2 + S$ for which $|x_2 - \mu_2| < \varepsilon$ for some $\varepsilon > 0$. The restriction in Theorem 7 that the conditional distribution of X_1 given X_2 be nondegenerate is necessary: If R in (2) has a mass $F(0)$ at zero with $0 < F(0) < 1$ and a density of the form (28) on $(0, \infty)$, then the conditional distribution of X_1 given X_2 is normal with probability 1, but X is not normally distributed. Theorem 7 was obtained by Kelker [3] under the assumptions that Σ is nonsingular and X has a density.

(c) We shall refer to the function $g_n(\cdot)$, appearing in (23) as the functional form of the density of $EC_n(\mu, \Sigma, \phi)$, whenever it exists, and we shall show that the normal distributions are the only elliptically contoured distributions for which marginal densities of different dimensions have the same functional form.

THEOREM 8. $EC_n(\mu, \Sigma, \phi)$ with $r(\Sigma) \geq 2$ is a normal distribution if and only if two marginal densities of different dimensions exist and have functional forms which agree up to a positive multiple.

Proof. The "only if" part is obvious, so we shall only show the "if" part. Suppose $X \sim EC_n(\mu, \Sigma, \phi)$ has marginals of dimensions p and $p+q$ with functional forms g_p and g_{p+q} , and

$$g_{p+q}(u) = (\text{const}) g_p(u), \quad u \geq 0. \quad (29)$$

(Here and below, "const" stands for a positive constant, not always the same.) Without loss of generality we assume that $r(\Sigma) = n$ and, in fact, that $\mu = 0$ and $\Sigma = I_n$. (Otherwise, we could consider $Y = (X - \mu)A^{-1} \sim EC_k(0, I_k, \phi)$, where $k = r(\Sigma)$ and $A'A$ is a rank factorization of Σ .) Then

$$\begin{aligned} g_p(x_1^2 + \cdots + x_p^2) &= \int_{\mathbb{R}^q} g_{p+q}(x_1^2 + \cdots + x_{p+q}^2) dx_{p+1} \cdots dx_{p+q} \\ &= (\text{const}) \int_{\mathbb{R}^q} g_p(x_1^2 + \cdots + x_{p+q}^2) dx_{p+1} \cdots dx_{p+q}, \end{aligned}$$

which implies

$$g_p(u) = (\text{const}) \int_{\mathbb{R}^q} g_p(u + z_1^2 + \cdots + z_q^2) dz_1 \cdots dz_q, \quad u \geq 0.$$

It follows that

$$g_{p+q}(x_1^2 + \cdots + x_{p+q}^2) = (\text{const}) \int_{\mathbb{R}^q} g_p(x_1^2 + \cdots + x_{p+2q}^2) dx_{p+q+1} \cdots dx_{p+2q}.$$

Thus, a multiple of $g_p(x_1^2 + \cdots + x_{p+2q}^2)$ is a density of some $Z \sim E_{p+2q}(0, I_{p+2q}, \psi)$ (see the third paragraph of Section 4), and Z has the $(p+q)$ -dimensional marginal density $g_{p+q}(x_1^2 + \cdots + x_{p+q}^2)$. Consequently, $\phi = \psi \in \Phi_{p+2q}$. Similarly, it follows that $\phi \in \Phi_{p+jq}$ for all $j = 1, 2, \dots$. Hence $\phi \in \Phi_\infty$, and there exists a distribution function F_∞ on $[0, \infty)$ such that both g_p and g_{p+q} satisfy (24). By the uniqueness of Laplace transforms and (29), we obtain $r^{-p} dF_\infty(r) = (\text{const}) r^{-(p+q)} dF_\infty(r)$, from which it follows that F_∞ is degenerate at some point $\sigma > 0$. Thus, $g_p(u) = \exp[-u/(2\sigma^2)]$ and X is normal. (Observe that in (29) the constant must be unity.) ■

(d) Because of the one-to-one correspondence between ϕ and F in (1), shown in Theorem 2, it is easily seen from Theorem 6 that, *under the assumptions of Theorem 6, the conditional distribution of $R_{q(X_2)}$ (defined in Corollary 5) given X_2 is independent of X_2 if and only if X is normal*. Thus, when X is normal, every conditional moment of $R_{q(X_2)}$ given X_2 is nonstochastic (degenerate). We now show that this property characterizes normality:

COROLLARY 8a. *Suppose $X = (X_1, X_2) \sim EC_n(\mu, \Sigma, \phi)$ and the ranks k_1 and k_2 appearing in Corollary 5 are strictly positive. Then, for any fixed positive integer p , $\mathcal{S}((R_{q(X_2)})^p | X_2)$ is finite and degenerate (i.e., independent of X_2) if and only if X is normally distributed.*

Proof. Just the “only if” part requires proof. It follows from the proof of Corollary 5 that the assumption “ $\mathcal{S}((R_{q(X_2)})^p | X_2)$ is finite and degenerate” is equivalent to “ $\mathcal{S}(\|Z_1\|^p | Z_2)$ is finite and degenerate,” where $Z = (Z_1, Z_2) \sim EC_k(0, I_k, \phi)$ with Z_1 and Z_2 of dimensions k_1 and k_2 , respectively. Moreover, if Z is normal, so is X . Thus, without loss of generality, we may assume $X = (X_1, X_2) \sim EC_n(0, I_n, \phi)$, where X_1 is of dimension m ($1 \leq m < n$) as in Theorem 5, and show that X is normal when $\mathcal{S}(\|X_1\|^p | X_2)$ is finite and degenerate. The latter permits two cases: $P(X=0)=1$ and $P(X=0)=0$. In the first case X is degenerate normal. In the second case, $P(\|X_2\|=0)=0$, and according to (12) and (15), $\mathcal{S}(\|X_1\|^p | X_2)=0$ when $F(\|X_2\|)=1$; and when $F(\|X_2\|)<1$,

$$\mathcal{S}(\|X_1\|^p | X_2) = \frac{\int_{(\|X_2\|, \infty)} (r^2 - \|X_2\|^2)^{(m+p)/2-1} r^{-(n-2)} dF(r)}{\int_{(\|X_2\|, \infty)} (r^2 - \|X_2\|^2)^{m/2-1} r^{-(n-2)} dF(r)},$$

where F is the d.f. of R in $X \stackrel{d}{=} RU^{(n)}$. Thus, the above ratio of integrals is a (nonrandom) constant a.s., which implies

$$\begin{aligned} & \int_{(u, \infty)} (r^2 - u^2)^{(m+p)/2-1} r^{-(n-2)} dF(r) \\ &= (\text{const}) \int_{(u, \infty)} (r^2 - u^2)^{m/2-1} r^{-(n-2)} dF(r) \end{aligned}$$

for almost every $u > 0$ (Leb.) since $\|X_2\|$ is absolutely continuous on $(0, \infty)$ with (according to (8)) a positive density at every $u > 0$ for which $F(u) < 1$. It follows that

$$\begin{aligned} & \int_{(u, \infty)} (r^2 - u^2)^{(m+p)/2-1} r^{-(n+p-2)} dF_{n+p}(r) \\ &= (\text{const}) \int_{(u, \infty)} (r^2 - u^2)^{m/2-1} r^{-(n+p-2)} dF_{n+p}(r) \end{aligned} \quad (30)$$

for almost every $u > 0$, where $F_{n+p}(p) = \int_{(0, p]} r^p dF(r) / \int_0^\infty r^p dF(r)$. ($\int_0^\infty r^p dF(r)$ is positive because $P(X=0)=0$ and finite because $\mathcal{E} \|X_1\|^p = \mathcal{E} [\mathcal{E}(\|X_1\|^p | X_2)] = \mathcal{E}(\|X_1\|^p | X_2)$ a.s.). It is apparent from (21) that the left and right integrals of (30) are functional forms for marginal densities of $Z \sim EC_{n+p}(0, I_{n+p}, \psi)$ of dimensions $n-m$ and $n-m+p$, respectively, where ψ is defined by the right side of (1) with $k=n+p$ and integrator F_{n+p} . Hence by Theorem 8, Z is normal. Thus, F_{n+p} is the d.f. of a multiple of a chi-variable with $n+p$ degrees of freedom. This implies F is the d.f. of a multiple of a chi-variable with n degrees of freedom, and, thus, X is normal. ■

Corollary 8a should hold for any real $p > 0$, not necessarily an integer. However, the size and nature of the class of functions $h: [0, \infty) \rightarrow \mathbb{R}$ for which " $\mathcal{E}(h(\|X_1\|) | X_2)$ is finite and degenerate" implies " X is normal" is not currently known.

(e) When $X = (X_1, X_2)$ is normal, the conditional central moments of all orders of X_1 given X_2 are finite and independent of X_2 . This property characterizes normality:

COROLLARY 8b. Suppose $X = (X_1, X_2) \sim EC_n(\mu, \Sigma, \phi)$, the ranks k_1 and k_2 appearing in Corollary 5 are strictly positive, and p is a positive integer. Then a p th-order conditional central moment of X_1 given X_2 is finite and degenerate (i.e., independent of X_2) if and only if X is normally distributed.

Proof. Just the "only if" part requires proof. Putting in (18), $X_1 = (X^1, \dots, X^m)$, $\mu_{X_2} = (\mu^1, \dots, \mu^m)$, and $A^* = (a_1, \dots, a_m)$, where a_i is the i th column of A , we have for $p_1 + \dots + p_m = p$,

$$((X^1 - \mu^1)^{p_1} \dots (X^m - \mu^m)^{p_m} | X_2) \stackrel{d}{=} (R_{q(X_2)})^p (U^{(k_1)})^p a_1^{p_1} \dots a_m^{p_m}.$$

Since $R_{q(X_2)}$ and $U^{(k_1)}$ are independent, the normality of X is immediate from Corollary 8a. ■

(f) The following, while not a characterization of normality, is an interesting extension of Theorem 6.

THEOREM 9. *Let $X = (X_1, X_2) \sim EC_n(\mu, \Sigma, \phi)$ and assume the ranks k_1 and k_2 appearing in Corollary 5 are strictly positive. Then*

$$\phi(u) = \int_{[0, \infty)} \exp(-ur^2/2) dG(r), \quad u \geq 0, \quad (31)$$

for some distribution function G on $[0, \infty)$ if and only if

$$\phi_{q(X_2)}(u) = \int_{[0, \infty)} \exp(-ur^2/2) dG_{q(X_2)}(r), \quad u \geq 0, \text{ a.s.}, \quad (32)$$

where G_{a^2} is a distribution function on $[0, \infty)$ given by

$$\begin{aligned} G_0(\rho) &= 1, & \rho &\geq 0, \\ G_{a^2}(\rho) &= \frac{\int_{(0, \rho]} r^{-k_2} \exp[-a^2/(2r^2)] dG(r)}{\int_{(0, \infty)} r^{-k_2} \exp[-a^2/(2r^2)] dG(r)}, & a > 0. \end{aligned} \quad (33)$$

Proof. By Lemma 4 below, G is related to the d.f. F of R , appearing in the full rank representation (2) of X , as follows:

$$\begin{aligned} F(0) &= G(0), \\ f(r) &= 2^{-k_2/2+1} (\Gamma(k_2/2))^{-1} r^{k_2-1} \int_{(0, \infty)} \rho^{-k_2} \exp[-r^2/(2\rho^2)] dG(\rho), \quad r > 0, \end{aligned} \quad (34)$$

where f is the density of F on $(0, \infty)$ on which it is absolutely continuous. If F_{a^2} denotes the d.f. of R_{a^2} appearing in Corollary 5, then it follows from (15) that

$$\begin{aligned} F_0(0) &= 1, \\ f_{a^2}(r) &= \frac{r^{k_1-1} \int_{(0, \infty)} \rho^{-k_2} \exp[-(r^2 + a^2)/(2\rho^2)] dG(\rho)}{2^{k_1/2-1} \Gamma(k_1/2) \int_{(0, \infty)} \rho^{-k_2} \exp[-a^2/(2\rho^2)] dG(\rho)}, \quad a > 0, \\ & \quad r \geq 0, \end{aligned}$$

where f_{a^2} denotes the density of F_{a^2} (which is absolutely continuous when $a > 0$). Again, by Lemma 4, the latter is equivalent to (32) and (33). ■

LEMMA 4. *Suppose $X \sim EC_n(\mu, \Sigma, \phi)$ has the full rank representation*

$X = {}^d \mu + RU^{(k)}A$. Then ϕ is given by (31) if and only if G is related to the distribution function F of R as described in (34).

Proof. Now $RU^{(k)} \sim EC_k(0, I_k, \phi)$. If ϕ is given by (31), then $RU^{(k)} = {}^d YZ$ where Y and Z are independent, Y has d.f. G , and $Z \sim N_k(0, I_k)$. Setting $W = \|Z\|$, we have $R = {}^d YW$, which implies

$$F(r) = G(0) + \int_{(0, \infty)} f_W(r/\rho) dG(\rho), \quad r > 0,$$

where f_W is the density of W , and the result follows from $f_W(r) = 2^{-(k/2)+1}(\Gamma(k/2))^{-1} r^{k-1} \exp(-r^2/2)$, $r > 0$. ■

REFERENCES

- [1] DAS GUPTA, S., EATON, M. L., OLKIN, I., PERLMAN, M., SAVAGE, L. J., AND SOBEL, M. (1972). Inequalities on the probability content of convex regions for elliptically contoured distributions. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* 2, pp. 241–264. Univ. of California Press, Berkeley.
- [2] FELLER, W. (1971). *An Introduction to Probability and Its Applications*, vol. 2., 2nd ed. Wiley, New York.
- [3] KELKER, D. (1970). Distribution theory of spherical distributions and a location-scale parameter generalization. *Sankhyā Ser. A* 32, 419–430.
- [4] LUKACS, E. (1970). *Characteristic Functions*. Hafner, New York.
- [5] SCHOENBERG, I. J. (1938). Metric spaces and completely monotone functions. *Ann. of Math.* 39 811–841.